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Continuity Theorems for the Rational Product Approximation Operator

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This paper establishes that the rational product approximation operator is continuous. Under an additional restriction, this operator satisfies a local Lipschitz condition.

1. INTRODUCTION

A number of recent papers [2, 3, 6–8, 12, 13] have considered extensions of the concept of *product approximation*. In this paper, *rational product approximation* is considered. In particular, this paper is the rational companion to a recent paper [7] in which continuity questions for linear product approximation were investigated. A brief description of rational product approximation and the rational product approximation operator follows.

Designate by $\mathbf{C}$ the set of all parameters consisting of the zero vector of E_{n+m+1} and all vectors

$$C = (A; B) = (a_0, a_1, \ldots, a_n; b_1, \ldots, b_m) \in E_{n+m+1}$$

satisfying

(i) at least one $|a_i| > 0$,

(ii) $P(A, x) = a_0 + a_1 x + \cdots + a_n x^n$ and $Q(B, x) = 1 + b_1 x + \cdots + b_m x^m$ have no common factors, and

(iii) $Q(B, x) > 0$ on $I = [-1, 1]$.

Then $\mathbf{R}(n, m)$ is defined to be the set of all rational functions $R(C, x) = P(A, x)/Q(B, x)$ with coefficient vectors $C = (A; B) \in \mathbf{C}$.

Let $D = I \times J = [-1, 1] \times [-1, 1]$. If $F \in C(D)$, then for each $y \in J$ define $F_y \in C(I)$ by $F_y(x) = F(x, y)$. Let

$$\begin{aligned} R(C_F(y), \cdot) &= P(A_F(y), \cdot)/Q(B_F(y), \cdot), \\ C_F(y) &= (a_0^F(y), \ldots, a_n^F(y); b_1^F(y), \ldots, b_m^F(y)) \in \mathbf{C} \end{aligned} \tag{1.1}$$

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be the best approximation to F_y from $\mathbf{R}(n, m)$ in the sense of the uniform norm $\|\cdot\|_I$ on I. If each $a_i^F(y)$ and $b_j^F(y)$ is continuous on J, then let

$$T_{r_i} a_i^F(y) = \sum_{k=0}^{r_i} a_{ik}^F y^k$$

be the best approximation to $a_i^F(\cdot)$ by polynomials of degree less than or equal to r_i in the sense of the uniform norm $\|\cdot\|_J$ on J, and let

$$T_{s_j} b_j^F(y) = \sum_{k=0}^{s_j} b_{jk}^F y^k$$

be the best uniform approximation to $b_j^F(y)$ on J by polynomials of degree less than or equal to s_j. Since $Q(B_F(y), x) > 0$ on D, for sufficiently large s_j,

$$1 + \sum_{j=1}^{m} T_{s_j} b_j^F(y)\, x^j > 0$$

on D. In this case,

$$\begin{aligned} R(F, x, y) &= \frac{\sum_{i=0}^{n} T_{r_i} a_i^F(y)\, x^i}{1 + \sum_{j=1}^{m} T_{s_j} b_j^F(y)\, x^j} \\ &= \frac{\sum_{i=0}^{n} \sum_{k=0}^{r_i} a_{ik}^F y^k x^i}{1 + \sum_{j=1}^{m} \sum_{k=0}^{s_j} b_{jk}^F y^k x^j} \end{aligned} \tag{1.2}$$

is defined to be the rational product approximation to F on D. The rational product approximation operator is then defined for $F \in C(D)$ by

$$(\mathscr{R}F)(x, y) = R(F, x, y), \qquad (x, y) \in D,$$

where $R(F, \cdot, \cdot)$ is the rational product approximation (1.2) to F on D. We note that $\mathscr{R}$ is a mapping of a subset of $C(D)$ into an appropriate class of rational functions (depending on the integers n, m, $\{r_i\}_{i=0}^{n}$, and $\{s_j\}_{j=1}^{m}$) contained in $C(D)$.

The lack of unicity for best approximations in several variables has posed difficulties in computation and in establishing results corresponding to classical theory of one dimensional approximation. Because of this Weinstein [12, 13] formally devised the concept of linear product approximation. Brown, M. Henry, J. Henry, and Weinstein [2, 3, 6, 8] then considered nonlinear product approximation. These papers discuss existence and computations. The authors of the present paper established a continuity theorem and a Lipschitz theorem for the linear product approximation operator [7]. In this paper, similar theorems are established for the more complex setting of rational product approximation.

2. Continuity of $\mathscr{R}$

In this section, we establish that the operator $\mathscr{R}$ is defined and continuous on an appropriate subset of $C(D)$. The following univariate results are needed in this section.

For $R(C, \cdot) = P(A, \cdot)/Q(B, \cdot) \in \mathbf{R}(n, m)$, define (as in [10, p. 79]) the degree of R at C to be

$$\begin{aligned} m(C) &= 1 + \max\{n + \partial Q, m + \partial P\}, & R &\not\equiv 0, \\ &= 1 + n, & R &\equiv 0, \end{aligned}$$

where ∂P and ∂Q are the degrees of the polynomials P and Q, respectively. Let

$$\mathbf{C}^* = \{(A; B) \in \mathbf{C}: a_n \neq 0 \text{ or } b_m \neq 0\}$$

and define

$$\mathbf{R}^* = \{R(C, \cdot) \in \mathbf{R}(n, m): C \in \mathbf{C}^*\}.$$

It can be shown that $\mathbf{R}^*$ is an open subset of $\mathbf{R}(n, m)$ (see the proof of Theorem 7-1 of Rice [11, p. 5]).

For $f \in C(I)$, let $C_f = (A(f), B(f)) \in \mathbf{C}$ be such that $R(C_f, \cdot)$ is the best uniform approximation to f from $\mathbf{R}(n, m)$ on I. If $C_f \in \mathbf{C}^*$, then f is said to be *normal.*

The following theorem, due to Brown and Henry [2], and a series of lemmas precede the principal theorem of this section.

Theorem 1. *If $F \in C(D)$ and F_y is normal for all $y \in J$, then $C_F(y)$ is continuous on J.*

The next lemma asserts that the collection of functions $F \in C(D)$ satisfying the hypotheses of Theorem 1 is open.

Lemma 1. *Suppose $F \in C(D)$ and F_y is normal for all $y \in J$. Then there is a $\delta = \delta(F) > 0$ such that $G \in C(D)$ and $\| G - F \|_D < \delta$ ensures that G_y is normal for every $y \in J$.*

Proof. Assume this is not the case. Then there is a sequence $\{G^k\}$ in $C(D)$ where $\| G^k - F \|_D \to 0$ and where for each k there is a $y_k \in J$ for which $G^k_{y_k}$ is not normal. Since J is compact we may assume $y_k \to y^* \in J$. The triangle inequality then implies that $\| G^k_{y_k} - F_{y^*} \|_I \to 0$. Then by the continuity of the best univariate rational approximation operator $R(C_{G^k}(y_k), \cdot) \to R(C_F(y^*), \cdot)$. Since $R(C_F(y^*), \cdot) \in \mathbf{R}^*$ and $\mathbf{R}^*$ is an open subset of $\mathbf{R}(n, m)$, eventually $R(C_{G^k}(y_k), \cdot) \in \mathbf{R}^*$, a contradiction.

For the remainder of this section we assume that $F \in C(D)$ and F_y is normal for all $y \in J$. Define

$$\rho(y) = \| F_y - R(C_F(y), \cdot)\|_I .$$

It follows from the continuity of the best univariate rational approximation operator that $\rho(y)$ is continuous on J. For $C = (A; B)$ and $\bar{C} = (\bar{A}; \bar{B})$ elements of E_{n+m+1}, define

$$\sigma(C; \bar{C}) = \max_{\substack{0 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} \{| a_i - \bar{a}_i |, | b_j - \bar{b}_j |\}.$$

LEMMA 2. *Given $\epsilon > 0$ there is a $\delta = \delta(F, \epsilon) > 0$ such that if $C(y) = (A(y); B(y)) = (a_0(y),..., a_n(y); b_1(y),..., b_m(y)) \in \mathbf{C}$ for each $y \in J$ and*

$$\| F_y - R(C(y), \cdot)\|_I \leqslant \rho(y) + \delta$$

for all $y \in J$, then $\sigma(C_F(y), C(y)) < \epsilon$ for all $y \in J$.

Proof. Assume otherwise. Then there is a sequence $\{C^k(y)\}_{k=1}^{\infty}$, $C^k(y) = (A^k(y); B^k(y)) = (a_0^k(y),..., a_n^k(y); b_1^k(y),..., b_m^k(y))$, where $C^k(y) \in \mathbf{C}$ for all $y \in J$, such that

$$\| F_y - R(C^k(y), \cdot)\|_I \leqslant \rho(y) + 1/k \tag{2.1}$$

for all $y \in J$, and there is a $y_k \in J$ for which

$$\sigma(C_F(y_k), C^k(y_k)) \geqslant \epsilon. \tag{2.2}$$

Again we may assume $y_k \to y^* \in J$.

Let $P^k = P(A^k(y_k), \cdot), \cdot)$, $Q^k = Q(B^k(y_k), \cdot)$, $P^* = P(A_F(y^*), \cdot)$, $Q^* = Q(B_F(y^*), \cdot)$, $w_k = \| P^k \|_I + \| Q^k \|_I$, and $w^* = \| P^* \|_I + \| Q^* \|_I$. Define $N^k = P^k/w_k$, $D^k = Q^k/w_k$, $N^* = P^*/w^*$, and $D^* = Q^*/w^*$. Since $\| D^k \|_I \leqslant 1$, by appropriate relabeling we may assume $D^k \to \bar{D}$. Similarly, $N^k \to \bar{N}$, and $\| \bar{N} \|_I + \| \bar{D} \|_I = 1$. Let $M = \| F \|_D + \max_J | \rho(y)| + 1$. From (2.1) we have

$$\| N^k/D^k \|_I = \| P^k/Q^k \|_I \leqslant M.$$

So, $| N^k(x)| \leqslant M | D^k(x)|$ for each $x \in I$. Thus

$$| \bar{N}(x)| \leqslant M | \bar{D}(x)|. \tag{2.3}$$

This inequality and $\| \bar{N} \|_I + \| \bar{D} \|_I = 1$ imply $\bar{D} \not\equiv 0$. Thus, using (2.3), we may perform appropriate cancellations to obtain an $N'/D' \in \mathbf{R}(n, m)$ such that

$$\bar{N}(x)/\bar{D}(x) = N'(x)/D'(x), \tag{2.4}$$

where $\bar{D}(x) \neq 0$. Thus at all but the finitely many points where $\bar{D}$ vanishes,

$$\begin{aligned}\left| F_{y^*}(x) - \frac{N'(x)}{D'(x)} \right| &= \left| F_{y^*}(x) - \frac{\bar{N}(x)}{\bar{D}(x)} \right| \\ &\leqslant \| F_{y^*} - F_{y_k} \|_I + \| F_{y_k} - R(C^k(y_k), \cdot) \|_I \\ &\quad + \left| \frac{N^k(x)}{D^k(x)} - \frac{\bar{N}(x)}{\bar{D}(x)} \right| \\ &\leqslant \| F_{y^*} - F_{y_k} \|_I + \rho(y_k) + 1/k + \left| \frac{N^k(x)}{D^k(x)} - \frac{\bar{N}(x)}{\bar{D}(x)} \right|.\end{aligned}$$

Letting $k \to \infty$, we get

$$\left| F_{y^*}(x) - \frac{N'(x)}{D'(x)} \right| \leqslant \rho(y^*).$$

By the continuity of $F_{y^*} - (N'/D')$,

$$\| F_{y^*} - (N'/D') \|_I \leqslant \rho(y^*).$$

By the uniqueness of best rational approximations,

$$N'/D' = R(C_F(y^*), \cdot) = R^* = P^*/Q^* = N^*/D^*.$$

These equations, (2.3), and (2.4) imply that $\bar{N} = R^* \bar{D}$, where $\bar{D}(x) \geqslant 0$ on I and $\| \bar{N} \|_I + \| \bar{D} \|_I = \| N^* \|_I + \| D^* \|_I$. Since F_{y^*} is normal, a lemma of Cheney ([4, p. 165]) implies $\bar{N} = N^*$ and $\bar{D} = D^*$. But $Q^*(0) = Q^k(0) = 1$, $Q^* = w^* D^*$, and $Q^k = w_k D^k$ imply that $D^*(0) = 1/w^*$ and $D^k(0) = 1/w^k$. Since $D^k(0) \to D^*(0)$, $w_k \to w^*$. Thus $P^k \to P^*$ and $Q^k \to Q^*$. As a consequence, $\sigma(C_F(y^*), C^k(y_k)) \to 0$. But (2.2) and the continuity of $C_F(\cdot)$ then imply $\sigma(C_F(y^*), C_F(y^*)) \geqslant \epsilon$, which is false.

LEMMA 3. *Given $\epsilon > 0$, there is a $\delta = \delta(F, \epsilon) > 0$ such that whenever $G \in C(D)$ and $\| G - F \|_D < \delta$, $\sigma(C_G(y), C_F(y)) < \epsilon$ for all $y \in J$.*

Proof. Denote $\beta(y) = \| G_y - R(C_G(y), \cdot) \|_I$. As in the proof of Theorem 1 of [7], it can be shown that $| \beta(y) - \rho(y) | \leqslant \| G - F \|_D$.

Let $\epsilon > 0$ be given. From Lemma 2, select $\delta > 0$ such that $G \in C(D)$ and $\| F_y - R(C_G(y), \cdot) \|_I \leqslant \rho(y) + \delta$ for all $y \in J$ implies $\sigma(C_F(y), C_G(y)) < \epsilon$ for all $y \in J$. Suppose $\| G - F \|_D < \delta/2$. Then for any $y \in J$, $\beta(y) - \rho(y) < \delta/2$ and

$$\begin{aligned}\| F_y - R(C_G(y), \cdot) \|_I &\leqslant \| F - G \|_D + \| G_y - R(C_G(y), \cdot) \|_I \\ &\leqslant \delta/2 + \beta(y) \\ &\leqslant \rho(y) + \delta.\end{aligned}$$

Thus $\sigma(C_G(y), C_F(y)) < \epsilon$ for all $y \in J$, completing the proof.

We now fix positive integers $r_0, ..., r_n, s_1, ..., s_m$. Since $Q(B_F(y), x) > 0$ on I for each $y \in J$ and $B_F(y)$ is continuous on J, $s_1, ..., s_m$ can be chosen large enough that

$$1 + \sum_{j=1}^{m} T_{s_j} b_j^F(y)\, x^j > 0$$

on D, where $T_{s_j} b_j^F$ is the best approximation to b_j^F on J by polynomials of degree less than or equal to s_j; see (1.2). Let

$$\tau = \min_D \left[1 + \sum_{j=1}^{m} T_{s_j} b_j^F(y)\, x^j \right].$$

LEMMA 4. *There is a* $\delta > 0$ *such that* $G \in C(D)$ *and* $\| G - F \|_D < \delta$ *ensures that*

$$1 + \sum_{j=1}^{m} T_{s_j} b_j^G(y)\, x^j \geqslant \tau/2$$

on D.

Proof. By the continuity of each T_{s_j}, there is a $\gamma > 0$ such that $\| b_j^F - b_j^G \|_J < \gamma$ implies that $\| T_{s_j} b_j^F - T_{s_j} b_j^G \|_J < \tau/2m$. Pick $\delta > 0$, via Lemma 3, so that $G \in C(D)$ and $\| G - F \|_D < \delta$ implies that $\| b_j^F - b_j^G \|_J < \gamma$, $j = 1, ..., m$. For such G,

$$\begin{aligned} 1 + \sum_{j=1}^{m} T_{s_j} b_j^G(y)\, x^j &\geqslant 1 + \sum_{j=1}^{m} T_{s_j} b_j^F(y)\, x^j - \sum_{j=1}^{m} \| T_{s_j} b_j^G - T_{s_j} b_j^F \|_J \\ &\geqslant \tau - \tau/2 \\ &= \tau/2. \end{aligned}$$

Remark. If we view the domain of $\mathscr{R}$ defined in (1.2) to be the set of all functions $F \in C(D)$ for which F_y is normal for all $y \in J$ and the denominator of (1.2) does not vanish on D, then Lemmas 1 and 4 imply that the domain of $\mathscr{R}$ is an open subset of $C(D)$.

We are now in a position to prove that the rational product approximation operator $\mathscr{R}$ is a continuous map from an appropriate open subset of $C(D)$ into a subclass of rational functions contained in $C(D)$.

THEOREM 2. *For* $F \in C(D)$ *let* F_y *be normal for each* $y \in J$. *Then* $\mathscr{R}$ *is continuous at F.*

Proof. In view of the above remark $\mathscr{R}$ is defined in a neighborhood of F. Let $\{G^k\}_{k=1}^{\infty} \subseteq C(D)$, and suppose $G^k \to F$. We may assume without loss of generality that $\mathscr{R}G^k$ is defined. By Lemma 3, $C_{G^k}(y) \to C_F(y)$ uniformly on J.

By the continuity of the best polynomial approximation operators $T_{r_i} a_i^{G^k} \to T_{r_i} a_i^F$ and $T_{s_j} b_j^{G^k} \to T_{s_j} b_j^F$ uniformly on J. Hence

$$\sum_{i=0}^{n} T_{r_i} a_i^{G^k}(y)\, x^i \to \sum_{i=0}^{n} T_{r_i} a_i^F(y)\, x^i$$

and

$$1 + \sum_{j=1}^{m} T_{s_j} b_j^{G^k}(y)\, x^j \to 1 + \sum_{j=1}^{m} T_{s_j} b_j^F(y)\, x^j$$

uniformly on D. Moreover, by Lemma 4,

$$1 + \sum_{j=1}^{m} T_{s_j} b_j^{G^k}(y)\, x^j \geqslant \tau/2$$

for sufficiently large k. Hence $\mathscr{R}G^k \to \mathscr{R}F$ uniformly on D. That is, $\mathscr{R}$ is continuous at F.

We conclude this section by stating two theorems. The first is a uniform Lipschitz theorem for univariate rational approximation and the second asserts that the rational product approximation operator $\mathscr{R}$ satisfies a local Lipschitz condition for certain functions $F \in C(D)$.

THEOREM 3. *Suppose $\Gamma \subseteq C(I)$ is compact, f is normal for all $f \in \Gamma$, and that $\Gamma \cap \mathbf{R}(n, m) = \varnothing$. Then there exists a $\lambda_\Gamma > 0$ such that*

$$\| R(C_f, \cdot) - R(C_g, \cdot) \|_I \leqslant \lambda_\Gamma \| f - g \|_I$$

for all $f \in \Gamma$ and $g \in C(I)$.

Theorem 3 is a rational counterpart to similar results for linear univariate approximation (see [1, 5, 7]), and is the main ingredient needed to establish the next theorem. Although more complex, the proof of this last theorem is similar to that of Theorem 4 in [7], and hence is omitted.

THEOREM 4. *Suppose that $F \in C(D)$, F_y is normal for each $y \in J$, and $F_y \notin \mathbf{R}(n, m)$ for each $y \in J$. Then there are constants $\delta_F > 0$ and $\lambda_F > 0$ such that if $G \in C(D)$ and $\| G - F \|_D < \delta_F$, then*

$$\| \mathscr{R}G - \mathscr{R}F \|_D \leqslant \lambda_F \| G - F \|_D .$$

3. CONCLUSIONS

The results of this paper establish that the rational product approximation operator is defined on an appropriate open subset of $C(D)$ and is continuous on this subset, and if F satisfies an additional requirement $\mathscr{R}$ satisfies a local

Lipschitz condition at F. Results of this type are not possible in the usual setting of multivariate rational approximation (surface approximation).

Although sharp error estimates for rational product approximation have not been established to date, J, Henry [6] has shown in fairly general circumstances that rational product approximation is competitive computationally with other known techniques (cf. [9]) for computing rational approximations to multivariate functions. Thus it would appear that rational product approximation has the advantage of a theory paralleling the classical univariate theory and yet remains, at least in certain circumstances, computationally competitive with surface rational approximation. More research is needed in this direction.

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